Z. F. Thomsen

\[ G/B \rightarrow G = \text{semisimple lin alg gp} \]

\[ e.g.: \text{SL}_n \]

\[ B = \text{Borel subgp}, e.g. \begin{pmatrix} 1 & \cdot \\ 0 & 1 \end{pmatrix} \]

\[ w \in W = \text{Weyl gp} \Rightarrow \chi(w) = BwB/B \subseteq G/B \]

\[ \text{Schubert variety} \]

\[ \chi(w) \neq \emptyset \]

Natural questions:
- normal?
- CM?
- nat. sing?
- \( F \)-regular?

Demazure ('74) char. formula

Gap due in '83, equiv to normality

Seshadri '87 gave a char. indep proof

Other pfs: Anderson, Joseph \( \frac{1}{2} '85 \)

Ramanan-Ramanathan using \( F \)-splitting

Kumar '87: Demazure char. formula in

Kac-Moody setting (char \( \mathbb{R} = 0 \))
Rough structure of proof:

- Grauert - Riemenschneider (char $k=0$)
- Kumar vanishing result on BSDH variety
  - normal, CM, rat., sing.

Today:

- Look at $F$-splitting of BSDH varieties which then gives the Kumar vanishing.
  - Idea of Kempf: from vanishing get results on singularities.

BSDH (Bott - Samelson - Demazure - Hansen) variety

Write $w = s_{i_1} \ldots s_{i_k}$ in a min way
  - simple reflections in $W$

Have corresponding parabolic subgr.

$P_{i_1}, \ldots, P_{i_k} \supset B$

$\omega = [s_{i_1}, \ldots, s_{i_k}] \sim \mathbb{Z}(\omega)$

$\mathcal{Z} = \mathbb{Z}(\omega) = P_{i_1}B \times \ldots \times P_{i_k}B / B$

$\pi$

$X = X(\omega) = P_{i_1} \ldots P_{i_k} / B$

Fact: $\pi$ is rational resolution, i.e.
i) \( \pi = \text{birational} \)

ii) \( Z \) nonsingular

iii) \( \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_Z \)

iv) \( H^i_{\pi_*} \left\{ \begin{array}{ll}
\mathcal{O}_Z & = 0, \quad i > 0 \\
\mathcal{O}_X & \end{array} \right. \)

(This \( \Rightarrow X \) normal, CM, not. sing.)

cond i) follows from Bruhat decomp.

\[ \begin{array}{c}
\mathcal{O}_Z \\
\downarrow \pi^\prime-\text{bdle} \\
\mathcal{O}_X \\
\downarrow \pi^\prime-\text{bundle} \\
\end{array} \]

\( B \)

Kempf: \( 0 \rightarrow \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Z \rightarrow C \rightarrow 0 \)

Want: \( C = 0. \)

If \( \mathcal{L} \) line bdle on \( X \) that is \underline{ample}

Enough to prove: \( H^0(X, C \otimes \mathcal{L}^n) = 0 \)

for \( n \gg 0 \)

(since \( C \otimes \mathcal{L}^n \) glob. generated)
Conclusion: enough to show

\[ H^0(\mathbb{A}^n) \rightarrow H^0(\pi^*(\mathbb{A}^n)) \text{ isom} \]

\[ \text{for } n \gg 0 \]

\[ H^0(\mathbb{A}^n \otimes \pi^* O_Z) \]

( since \( H^1(\mathbb{A}^n) = 0 \) for \( n \gg 0 \)).

\[ Z(w) \hookrightarrow Z(w_0) \]

\[ \downarrow \]

\[ \downarrow \pi_0 \]

\[ X(w) \hookrightarrow G/B = X(w_0) \]

\[ \text{normal} \Rightarrow (\pi_0)_* O_{Z(w_0)} = O_{w_0} \]

Get \[ H^0(\pi^*(\mathbb{A}^n)) \leftarrow H^0(Z(w_0), \pi^*(\mathbb{A}^n)) \]

\[ J \leftarrow s^\uparrow \]

\[ H^0(\mathbb{A}^n) \leftarrow H^0(G/B, \mathbb{A}^n) \]

Hence it is enough to prove the top

horiz. restr. map is surjective for \( n \gg 0 \)

Kumar vanishing \( \Rightarrow \) this is OK if \( Z \)

is globally generated.

Kumar Vanishing:

\[ \omega_j = [s_i, \ldots, \hat{s}_i, \ldots, s_i] \hookrightarrow Z(\omega_j) = Z_{s_i} \]

\[ \text{divisor} \]

\[ Z(\omega_j) \]

\[ j = 1, \ldots, N \]
These "standard divisors" on $\mathbb{Z}$ generate $\text{Pic}(\mathbb{Z})$.

Let $\mathcal{M} = \text{glob. gen. line bundle on } Z(\omega)$
$1 \leq r \leq 2 \leq N$

Then
$H^i(Z(\omega), \mathcal{M} \otimes \mathcal{O}(-Z_r - \ldots - Z_2)) = 0$
for $i > 0$
$H^i(Z(\omega), \mathcal{M}) = 0 \quad i > 0$.

These vanishing imply the desired surjectivity: $Z(\omega) \to Z(\omega_0)$ can be described as a seq. of consecutive standard divisors:

$Z(\omega) \to \ldots \to \ldots Z(\omega_0)$
\[
\begin{array}{cc}
\text{std} & \text{div} \\
\text{std} & \text{div}
\end{array}
\]
This gives iii).

iv): Let $\mathcal{M} = \text{line bundle on } Z$
ample on $X$
$R^i \pi_* (\mathcal{M}) = 0$ iff $H^0(X, R^i \pi_* \mathcal{M} \otimes L^n) = 0$
for $n \gg 0$

Since $H^d(X, (R^i \pi_* \mathcal{M}) \otimes L^n) = 0$ for $i > 0$, $n \gg 0$

Leray spectral sequence $\Rightarrow$

$H^0(X, (R^i \pi_* \mathcal{M}) \otimes L^n) \cong H^i(Z, \mathcal{M} \otimes \pi^* L^n)$
If $M = \mathbb{Q}$, want to show:

$$H^i(\mathbb{Z}, \pi^* L^n) = 0 \quad i > 0, \quad n \gg 0$$

(from Kumar van, ok)

If $M = \omega_2$, and using

$$\omega_2 \cong \pi^* L(-g) \otimes O(-Z_1 - \ldots - Z_n)$$

$$\Rightarrow H^i(\mathbb{Z}, \omega_2 \otimes \pi^* L^n) = 0 \quad \text{for } i > 0, \quad n \gg 0$$

by Kumar vanishing.

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Global $F$-regularity (Karen Smith, 2000)

$X$ proj. variety, $L$ very ample line bundle

$s \in T(\mathfrak{x}, L) \setminus \{0\}$, $X$ globally $F$-reg if

- $\exists \varepsilon > 0$ s.t. $O_x \xrightarrow{\varepsilon} f_* O_x$ is split

the composition

and

- $X \setminus Z(s)$ $F$-regular

This $\Rightarrow$ in particular $X$ normal, CM, not sing.
\[ \text{Prop} \quad \text{BSDH - var. are globally } F\text{-regular.} \]

\[ \text{Prop} \quad \pi : Z \rightarrow X, \quad X, Z \text{ proj. var.} \]

\[ O_X \cong \pi_* O_Z \]

If \( Z = \text{globally } F\text{-reg} \)

\[ X \text{ is globally } F\text{-reg} \]

\[ \text{Cor. Schubert varieties are globally } F\text{-regular.} \]