Fundamental groups of algebraic varieties in positive characteristic

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Frobenius splitting, 2010
Outline

1. Known fundamental groups
   - Different approaches to algebraic fundamental groups
   - Characterizations of representations of $\pi^S_1(X, x)$

2. Properties of the S-fundamental group scheme
   - First properties
   - Lefschetz type theorems
   - $\pi^S_1$ for a product of varieties
   - Computation of the S-fundamental group
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   - Characterizations of representations of $\pi_1^S(X, x)$

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   - $\pi_1^S$ for a product of varieties
   - Computation of the S-fundamental group
Grothendieck’s fundamental group

$X$ a connected scheme
$x : \text{Spec } k \rightarrow X$ a geometric point
$T_x$ the fiber functor on the category of finite étale covers $Y \rightarrow X$ with objects $T_x(Y) = Y_x$

**Definition**

“Étale fundamental group” $\pi_1(X, x) = \text{the automorphism group of } T_x$.

Finite quotients of $\pi_1(X, x) = \text{finite étale Galois covers of } X$

$X/\mathbb{C}$

$\pi_1(X, x) = \hat{\pi}_1^\text{top}(X, x)$
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Tannaka duality

Neutral Tannaka category = (\(\mathcal{A}, \otimes, T : \mathcal{A} \to \text{Vect}_k, A\)).

\(\mathcal{A}\) an abelian category
\(\otimes\) tensor product
\(T : \mathcal{A} \to \text{Vect}_k\) a fiber functor
\(A \in \text{Ob} \mathcal{A}\) a trivial object

Tannaka duality theorem

Neutral Tannaka category (\(\mathcal{A}, \otimes, T : \mathcal{A} \to \text{Vect}_k, A\)) = category of representations of an affine \(k\)-group scheme \(G\).

\(G\) is called Tannaka dual to (\(\mathcal{A}, \otimes, T : \mathcal{A} \to \text{Vect}_k, A\)).
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Tannakian approach to fundamental group schemes

X proper integral $k$-scheme, $k = \bar{k}$

Vector bundle $E$ is finite if $\exists f \neq g \in k[x]$ s.t. $f(E) = g(E)$

Essentially finite = subquotient of finite

Definition

Nori’s fundamental group scheme $\pi_1^N(X, x)$ is the group scheme Tannaka dual to the neutral Tannaka category of essentially finite vector bundles on $X$

$\pi_1^N(X, x)$ classifies torsors under arbitrary finite group schemes
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Comparison of Grothendieck’s and Nori’s constructions:

- char 0: $\pi_1(X, x) = \pi_1^N(X, x)$
- char $p > 0$: $\pi_1(X, x)$ group of $k$-points of the maximal pro-étale quotient of $\pi_1^N(X, x)$
**S-fundamental group scheme**

\[ E \text{ numerically flat} \iff E \text{ locally free, } E, E^* \text{ nef} \]

\( \mathcal{C}^{\text{nf}}(X) \) numerically flat vector bundles

\( x \in X(k) \) fixed

\( T_x : \mathcal{C}^{\text{nf}}(X) \to \text{Vect}_k \) sends \( E \) to the fiber \( E(x) \).

**Definition**

**S-fundamental group scheme** \( \pi_1^S(X, x) \) is the group scheme Tannaka dual to the neutral Tannaka category

\( (\mathcal{C}^{\text{nf}}(X), \otimes, T_x, \mathcal{O}_X) \).

\( \pi_1^N(X, x) \) is a pro-finite completion of \( \pi_1^S(X, x) \).
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\( \pi_1^N(X, x) \) is a pro-finite completion of \( \pi_1^S(X, x) \)
Motivation: Simpson’s non-abelian Hodge theory

$X/\mathbb{C}$ smooth projective

- Complex representations of $\pi_{1}^{\text{top}}(X, x) = \text{semistable Higgs bundles with vanishing Chern classes}$
- Unitary representations of $\pi_{1}^{\text{top}}(X, x) = \text{semistable bundles with vanishing Chern classes}$

In other words: $\pi_{1}^{S}(X, x)$ is a pro-unitary completion of $\pi_{1}^{\text{top}}(X, x)$
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Local freeness of semistable sheaves.

**Theorem**

Let $X/k$ be smooth, $H$ ample, $E$ a strongly $H$-semistable, torsion free with vanishing Chern classes. Then $E$ locally free and

$$0 = E_0 \subset E_1 \subset ... \subset E_m = E,$$

$E_i/E_{i-1}$ stable, strongly semistable, locally free with vanishing Chern classes

For $n \gg 0$

$$0 = E'_0 \subset E'_1 \subset ... \subset E'_{m'} = (F^n)^*E,$$

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Theorem

$X$ smooth, projective. The following are equivalent for a coherent sheaf $E$:

1. $E$ locally free and numerically flat,
2. $E$ locally free, nef, degree 0 for some ample $H$,
3. $E$ reflexive, strongly $H$-semistable, $\text{ch}_1(E) \cdot H^{d-1} = 0$ and $\text{ch}_2(E) \cdot H^{d-2} = 0$,
4. $E$ torsion free, strongly $H$-semistable, $\chi(E) = r \chi(O_X)$, $\text{ch}_1(E) \cdot H^{d-1} = 0$ and $\text{ch}_2(E) \cdot H^{d-2} = 0$. 

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Fundamental groups of algebraic varieties in positive characteristic
Numerically flat vis-à-vis semistable

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For a smooth, projective variety $X$, the following are equivalent for a coherent sheaf $E$:

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Proof of the theorem

- $2 \Rightarrow 3$ follows from positivity of Chern classes for ample bundles (Fulton–Lazarsfeld)
- $4 \Rightarrow 1$ follows from boundedness of semistable sheaves with fixed numerical invariants
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   - First properties
   - Lefschetz type theorems
   - $\pi_1^S$ for a product of varieties
   - Computation of the S-fundamental group
• $f : X \rightarrow Y$ flat surjective, $f_* O_X = O_X \Rightarrow \pi^S_1(X, x) \rightarrow \pi^S_1(Y, y)$ faithfully flat
• $\pi^S_1(X, x) = 0$ for homogeneous $X$
• bad base change: $\pi^S_1(X_K, x) \neq \pi^S_1(X_k, x) \times_k K$
• Hogadi–Mehta 2010: $\pi^S_1(X, x)$ is a birational invariant
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Known fundamental groups
Properties of the S-fundamental group scheme
Summary

First properties
Lefschetz type theorems
$\pi_1^S$ for a product of varieties
Computation of the S-fundamental group

- $f : X \to Y$ flat surjective, $f_* \mathcal{O}_X = \mathcal{O}_X \Rightarrow$
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- \textit{Hogadi–Mehta} 2010: \( \pi_1^S(X, x) \) is a birational invariant
Vanishing theorems

$X$ of dimension $d$, $D$, $H$ ample divisors, $T_X(\alpha H)$ globally generated, $pD - \alpha H$ ample

- If $d \geq 2$ then for any $E \in C^{\rm{nf}}(X)$

$$H^1(X, E(-D)) = 0.$$  

- If $d \geq 3$ and

$$DH^{d-1} > \max \left( \alpha H^d, \frac{(d + 1)\alpha H^d - K_X H^{d-1}}{p} \right)$$

then $H^2(X, E(-D)) = 0$. 

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Fundamental groups of algebraic varieties in positive characteristic
Vanishing theorems

Let $X$ be a variety of dimension $d$, $D$, $H$ ample divisors, $T_X(\alpha H)$ globally generated, $pD - \alpha H$ ample.

1. If $d \geq 2$ then for any $E \in C^{\text{nf}}(X)$
   \[ H^1(X, E(-D)) = 0. \]

2. If $d \geq 3$ and
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Fundamental groups of algebraic varieties in positive characteristic
First Lefschetz theorem

**Theorem**

Let $D \subset X$ be an ample smooth effective divisor. If $d = \dim X \geq 2$ and

$$DH^{d-1} > \mu_{\max}(\Omega_X)$$

then $\pi_1^S(D, x) \to \pi_1^S(X, x)$ is faithfully flat.

**Caution:** Without assumption on $DH^{d-1}$ theorem is false.
First Lefschetz theorem

**Theorem**

\[ D \subset X \text{ ample smooth effective divisor. If } d = \dim X \geq 2 \text{ and } \]

\[ DH^{d-1} > \mu_{\text{max}}(\Omega_X) \]

then \( \pi_1^S(D, x) \rightarrow \pi_1^S(X, x) \) is faithfully flat.

**Caution:** without assumption on \( DH^{d-1} \) theorem is false
Second Lefschetz theorem

**Theorem**

Assume $d \geq 3$ and $T_X(\alpha H)$ globally generated for some $\alpha \geq 0$, $D \subset X$ smooth effective, $D - \alpha H$ ample.

If

$$DH^{d-1} > \max \left( p\alpha H^d, (d + 1)\alpha H^d - K_X H^{d-1} \right)$$

then $\pi_1^S(D, x) \simeq \pi_1^S(X, x)$. 

Theorem

Assume $d \geq 3$ and $T_X(\alpha H)$ globally generated for some $\alpha \geq 0$, $D \subset X$ smooth effective, $D - \alpha H$ ample.

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then $\pi_1^S(D, x) \simeq \pi_1^S(X, x)$. 
Vanishing theorems in presence of lifting

$X$ $d$-dimensional with lifting to $W_2(k)$.

Deligne–Illusie implies:

**Theorem**

If $D$ ample, $E \in C^\text{nf}(X)$. Then

$$H^i(X, E(-D) \otimes \Omega^i_X) = 0$$

if $i + j < \min(p, d)$. 
Lefschetz theorems in presence of lifting

Theorem

\( D \) smooth ample effective divisor on \( X \). \( X \) has a lifting to \( W_2(k) \)

1. If \( \dim X \geq 2 \) then \( \pi_1^S(D, x) \to \pi_1^S(X, x) \) is faithfully flat.

2. If \( \dim X \geq 3 \) and \( p \geq 3 \) then \( \pi_1^S(D, x) \cong \pi_1^S(X, x) \).
Theorem

Let $X, Y$ be complete varieties. Then

$$\pi^S_1(X \times_k Y) \simeq \pi^S_1(X) \times_k \pi^S_1(Y)$$

For Nori’s fundamental group scheme:

*Mehta–Subramanian 2002*
Theorem

$X, Y$ be complete varieties. Then

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$$

For Nori’s fundamental group scheme:

*Mehta–Subramanian 2002*
**Theorem**

Let $X, Y$ be smooth projective curves, and $F$ be locally free on $X \times Y$. Assume the following:

- $\forall x \in X \ F_x$ is semistable
- $F$ is numerically trivial

Then:

- $\forall y_1, y_2 \in Y \ F_{y_1} \simeq F_{y_2}$
- $\forall x \in X \ F_x$ are $S$-equivalent
Theorem

Let $X$, $Y$ be smooth projective curves, $F$ be a locally free sheaf on $X \times Y$. Assume

- $\forall x \in X \ F_x$ is semistable
- $F$ is numerically trivial

Then:

- $\forall y_1, y_2 \in Y \ F_{y_1} \cong F_{y_2}$
- $\forall x \in X \ F_x$ are $S$-equivalent
Abelian part of $\pi^S_1$

In positive characteristic:

$$\pi^S_{ab}(X, x) \simeq \lim \leftarrow \hat{G} \times \text{Diag}((\text{Pic}^\tau X)_{\text{red}}),$$

In characteristic zero:

$$\pi^S_{ab}(X, x) \simeq H^1(X, \mathcal{O}_X)^* \times \text{Diag}(\text{Pic}^\tau X).$$
Abelian part of $\pi_1^S$

In positive characteristic:

$$\pi_{ab}^S(X, x) \simeq \varprojlim \mathcal{G} \times \text{Diag}((\text{Pic}^\tau X)_{\text{red}}),$$

In characteristic zero:

$$\pi_{ab}^S(X, x) \simeq H^1(X, \mathcal{O}_X)^* \times \text{Diag}(\text{Pic}^\tau X).$$
Albanese morphism

\[ \text{alb}_X : X \to \text{Alb } X \text{ the Albanese morphism} \]

\[ 0 \to \lim_{G \subset \text{Pic}^0 X} \frac{G}{G_{\text{red}}} \times \text{Diag}(\text{NS}(X)_{\text{tors}}) \to \pi^S_{\text{ab}}(X) \to \pi^S_1(\text{Alb } X) \to 0 \]
Computation for simply connected varieties

Esnault–Mehta: If $\pi_1^N(X, x) = 0$ then $\pi_1^S(X, x) = 0$.

**Theorem**

Assume $\pi_1^{et}(X, x) = 0$, $E$ rank $r$ numerically flat. Then $\exists n \geq 0$ such that $(F_X^n)^* E \simeq \mathcal{O}_X^r$.

**Corollary**

If $\pi_1^{et}(X, x) = 0$ then $\pi^S(X, x) \simeq \pi_1^N(X, x)$. 
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**Corollary**

If $\pi_1^{\text{et}}(X, x) = 0$ then $\pi^S(X, x) \simeq \pi_1^N(X, x)$. 
There are interesting fundamental group schemes generalizing Grothendieck’s fundamental group.

S-fundamental group scheme encodes properties of strongly semistable bundles with vanishing Chern classes.

Outlook and possible applications in positive characteristic:

- Existence of a non-abelian Hodge theory
- Characterization of varieties with nef tangent bundle
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Summary

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For Further Reading

A. Langer

*On the S-fundamental group scheme.*

A. Langer

*On the S-fundamental group scheme II.*
preprint.

H. Esnault, V. Mehta

*Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles,*
to appear in *Inv. Math.*