Looking for compatibly split subschemes of $\text{Hilb}_n(\mathbb{A}^2_k)$

Jenna Rajchgot
(Supervised by Allen Knutson)

Department of Mathematics, Cornell University

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Let $k$ be an algebraically closed field of characteristic $p$.

**Definition:** As a set, define $\text{Hilb}_n(\mathbb{A}_k^2)$ to be:

$$\{ I \subset k[x,y] : \dim(k[x,y]/I) = n \text{ as a vector space over } k \}.$$ 

Each element $I \in \text{Hilb}_n(\mathbb{A}_k^2)$ corresponds to “$n$ points in the affine plane”.
More precisely, we have the following:

- If $I$ is a radical colength $n$ ideal, then $\text{Spec}(k[x, y]/I)$ consists of $n$ distinct closed points in the plane.

- If $k[x, y]/I$ is a local ring then $\text{Spec}(k[x, y]/I)$ is non-reduced and supported at a single point $(a, b)$.

For example, each of $\langle x^3, y \rangle$, $\langle x^2, xy, y^2 \rangle$, $\langle x^2 + y, xy, y^2 \rangle$ is an element of $\text{Hilb}_3(\mathbb{A}^2_k)$ corresponding to a triple point at the origin. There are many more ideals of this sort.

In fact, the family of all colength $n$ ideals supported at a given point $(a, b) \in \mathbb{A}^2_k$ is a subscheme of dimension $n - 1$.

- In general, $k[x, y]/I \cong k[x, y]/I_1 \times \cdots \times k[x, y]/I_r$ where each $k[x, y]/I_j$ is a local ring and the (vector space) dimensions of $k[x, y]/I_1, \ldots, k[x, y]/I_r$ sum to $n$. 
More on $\text{Hilb}_n(\mathbb{A}_k^2)$

**Fact:** $\text{Hilb}_n(\mathbb{A}_k^2)$ is a smooth, connected, integral scheme of dimension $2n$.

Even better than being reduced, it is Frobenius split. In particular, $\text{Hilb}_n(\mathbb{A}_k^2)$ is Frobenius split compatibly with the anticanonical divisor “at least one point is on the x-axis or at least one point is on the y-axis”.

The torus $T = (k^*)^2$ acts on $\text{Hilb}_n(\mathbb{A}_k^2)$ by scaling $x$ and $y$. That is, if $I \in \text{Hilb}_n(\mathbb{A}_k^2)$ and $I = \langle f_1(x, y), \ldots, f_d(x, y) \rangle$ then

$$(t_1, t_2) \cdot \langle f_1(x, y), \ldots, f_d(x, y) \rangle = \langle f_1(t_1x, t_2y), \ldots, f_d(t_1x, t_2y) \rangle.$$

The colength $n$ monomial ideals are the fixed points of this action.

Notice that the described compatibly split divisor of $\text{Hilb}_n(\mathbb{A}_k^2)$ is $T$-invariant. This will be useful later on.
A Question

One may now ask the following question:

What are all of the compatibly split subvarieties?
The compatibly split subvarieties of $\text{Hilb}_2(\mathbb{A}^2_k)$

Consider $\text{Hilb}_2(\mathbb{A}^2_k)$ along with the divisor, $D$, that determines the splitting.

One can intersect the irreducible components of this divisor and decompose the intersection to obtain some new compatibly split subvarieties.
Hilb₂(𝔸₂ᵦ) continued

These are not the only codimension 2 compatibly split subvarieties of Hilb₂(𝔸₂ᵦ). Neither irreducible component of D is R1. Therefore, the non-R1 loci should be included in the union of codimension 2 compatibly split subvarieties.

Intersecting each one of these (normal) subvarieties with the union of the others and then decomposing the intersections yields:

Repeating this procedure once more produces the T-fixed points.
Hilb$_2(A^2_k)$ continued

In this case, the above sequence of intersecting, decomposing and looking for non-$R1$ loci did in fact find all compatibly split subvarieties.

Allen Knutson (in unpublished work with Thomas Lam and David Speyer) made these ideas precise in the following algorithm.
An algorithm for finding compatibly split subvarieties

**Algorithm** (Knutson-Lam-Speyer)

*Input:* $(X, \partial X)$ where $X$ is Frobenius split and $\partial X$ is the anticanonical divisor which induces the splitting.

*Output:* Suppose that $\partial X = D_1 \cup \cdots \cup D_r$. Let $E_i = D_1 \cup \cdots \hat{D}_i \cup \cdots \cup D_n$. There are two cases.

1. If $X$ is regular in codimension 1, then return $(D_1, D_1 \cap E_1), \ldots, (D_n, D_n \cap E_n)$.

2. If $X$ is not $R1$, return $(\tilde{X}, \nu^{-1}(\partial X \cup X_{\text{non}\text{-}R1}))$ where $\nu : \tilde{X} \to X$ is the normalization of $X$.

Repeat until neither 1. nor 2. can be applied. When finished, map all subvarieties back to the original Frobenius split variety to obtain a list of many (for large $p$) compatibly split subvarieties.

At each stage of the algorithm, check if there is a component of the singular locus that is both compatibly split and of codimension $\geq 2$. (Hard!) If so, add it (and its compatibly split subvarieties) to the list. The final list consists of all compatibly split subvarieties of $(X, \partial X)$. 
Back to $\text{Hilb}_2(\mathbb{A}^2_k)$

As an example of the algorithm, consider (again) the case of $\text{Hilb}_2(\mathbb{A}^2_k)$.
Start with $(\text{Hilb}_2(\mathbb{A}^2_k), D)$ where $D$ is as before.

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
$$

Apply 1.

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix},
$$

Due to the symmetry, continue with just the first of the two pairs.
Next, recall that the components of $D$ are not regular in codimension 1. Apply 2.

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
$$
Apply 1.

Applying 1. once more obtains the preimage of the $T$-fixed points under the normalization map $\nu : X_n \to \text{Hilb}_2(\mathbb{A}^2_k)$ where $X_n$ denotes the isospectral Hilbert scheme (i.e. the scheme of *labelled* points in the affine plane).
The compatibly split subvarieties of $\text{Hilb}_2(\mathbb{A}^2_k)$
In the case of $\text{Hilb}_2(\mathbb{A}^2_k)$, repeatedly applying steps 1. and 2. was enough to find all compatibly split subvarieties. However, this is not always the case. The following example shows that, at each stage of the algorithm, one must check for compatibly split subvarieties inside of the singular locus.

**Example** Let $p \equiv 1 (\text{mod} 3)$ and let $f = x^3 + y^3 + z^3$ in $k[x, y, z]$. Then $\text{Tr}(f^p \cdot \cdot)$ is a splitting of $\mathbb{A}^3_k$ that compatibly splits $\{ f(x, y, z) = 0 \}$. In this case, $(X, \partial X) = (\mathbb{A}^3_k, \{ x^3 + y^3 + z^3 = 0 \})$. Neither 1. nor 2. can be applied. However, the origin is compatibly split.
A helpful tool

It was not very difficult to determine (and understand) the intersection of two compatibly split subvarieties of \( \text{Hilb}_2(\mathbb{A}^2_k) \). However, this becomes increasingly difficult as the number of points increases. To help overcome this difficulty, one can use the torus action on the Hilbert scheme.

Consider \( \text{Hilb}_2(\mathbb{C}^2) \).
Recall that \( T = (\mathbb{C}^*)^2 \) acts by scaling \( x \) and \( y \) with weights \((1, 0)\) and \((0, 1)\) respectively.

There is a moment map \( \Phi : \text{Hilb}_2(\mathbb{C}^2) \to \mathfrak{t}^* \). It is given by

\[
I \mapsto \sum_{i, j \in \mathbb{N} \cup \{0\}} d_{ij}(i, j)
\]

where (under a fixed Hermitian inner product), the rank \( n \) orthogonal projection of \( \mathbb{C}[x, y] \) to \( I^\perp \) is \( x^i y^j \mapsto d_{ij} x^i y^j + \cdots \).
Moment Polytopes

The image of the moment map is a convex polytope and the image (under the moment map) of the $T$-invariant subvarieties appear as subpolytopes.

Figure: Moment polytope for $\text{Hilb}_2(\mathbb{C}^2)$
Continued

Because the compatibly split divisor $D \subset \text{Hilb}_n(\mathbb{A}^2_k)$ is $T$-invariant, all other compatibly split subvarieties are as well. These subvarieties correspond to certain subpolytopes of the moment polytope. Thus, intersecting appropriate polytopes is helpful in understanding intersections of compatibly split subvarieties.

Note that even before applying the algorithm, one can use the polytope to determine a few compatibly split subvarieties. In particular, the preimage of each exterior face of the polytope is compatibly split.
Another polytope

Figure: Moment polytope for $\text{Hilb}_3(\mathbb{C}^2)$
The compatibly split subvarieties of $\text{Hilb}_3(\mathbb{A}_k^2)$
The moment polytope for $\text{Hilb}_4(\mathbb{A}^2_k)$
The compatibly split subvarieties of $\text{Hilb}_4(\mathbb{A}^2_k)$
A few remarks

▶ Since \(\{\langle x^2, y^2 \rangle\}\) is not compatibly split, it might be tempting to guess that, in general, the only compatibly split 0-dimensional subvarieties are the torus fixed points that map to exterior vertices of the moment polytope. One can see that this isn’t the case in \(\text{Hilb}_5(\mathbb{A}^2_k)\).

▶ Notice that the poset of compatibly split subvarieties for \(\text{Hilb}_{n_1}(\mathbb{A}^2_k)\) appears inside the poset for \(\text{Hilb}_{n_2}(\mathbb{A}^2_k)\) \((n_1 < n_2)\) by adding some more points.

▶ One can see from the algorithm that the subvariety of \(\text{Hilb}_n(\mathbb{A}^2_k)\) corresponding to “\(q\) points on the x-axis and \(r\) points on the y-axis” for \(q + r \leq n\) is compatibly split. The difficulty is in understanding the subvarieties where many points collide at the origin.
Thank You.