‘Frobenius Splittings’ conference

‘Graded annihilators and tight closure’

Rodney Y. Sharp

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Notation

Throughout, $R$ denotes a commutative Noetherian ring of prime characteristic $p$, and $f : R \longrightarrow R$ is the Frobenius homomorphism, so that $f(r) = r^p$ for all $r \in R$. 
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The Frobenius skew polynomial ring over $R$ is $R[x, f] := \bigoplus_{n \geq 0} R x^n$ (freely generated as left $R$-module by the powers $(x^n)_{n \geq 0}$ of the variable $x$) with $xr = r^p x$ for all $r \in R$. 
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A ‘Frobenius action’ on an $R$-module $H$ is a left $R[x, f]$-module structure on $H$ that extends its $R$-module structure.
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Always $H/\Gamma_x(H)$ is $x$-torsion-free.
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An element $r \in R$ belongs to $a^*$, the tight closure of $a$, if and only if there exists $c \in R^\circ$ such that $cr^p \in a^{[p^n]}$ for all $n \gg 0$.

Note that the element $c \in R^\circ$ is allowed to change as $r$ varies through $a^*$ and as $a$ varies.
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An element $r \in R$ belongs to $a^*$, the tight closure of $a$, iff $\exists c \in R^\circ$ such that $cr^p \in a[p^n]$ for all $n >> 0$.

Note that the element $c \in R^\circ$ is allowed to change as $r$ varies through $a^*$ and as $a$ varies.

A test element for ideals for $R$ is a $c' \in R^\circ$ such that, for every ideal $b$ of $R$ and every $r \in b^*$, we have $c'r^p \in b[p^n]$ for all $n \geq 0$. 
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$$rx^n a = r a^p x^n \quad \forall r, a \in R.$$
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So \( Rx^n \otimes_R (R/\alpha) \cong R/\alpha^{[p^n]} \).
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So a test element for ideals for \( R \) is a \( c \in R^\circ \) such that, for every ideal \( \mathfrak{b} \) of \( R \) and every \( r \in \mathfrak{b}^* \), for every \( n \geq 0 \), the element \( cx^n \) annihilates
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1 \otimes (r + \mathfrak{b}) \in Rx^0 \otimes_R (R/\mathfrak{b}) = (R[x, f] \otimes_R (R/\mathfrak{b}))_0.
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An element $m \in M$ belongs to $0^*_M$, the tight closure of 0 in $M$, $\iff \exists c \in R^\circ$ such that

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If $N$ is a submodule of $M$, then $N^*_M$, the tight closure of $N$ in $M$, is the inverse image of $0^*_M$ under the natural epimorphism $M \longrightarrow M/N$.  

– p. 7/38
Test elements for modules

A test element for modules for $R$ is a $c \in R^\circ$ such that, for every finitely generated $R$-module $M$, for every $m \in 0^*_M$, and for every $n \geq 0$, the element $cx^n$ annihilates $1 \otimes m \in Rx^0 \otimes R M = (R[x, f] \otimes R M)_0$. 
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Compare this with the definition of a test element for ideals for $R$ as a $c \in R^\circ$ such that, for every cyclic $R$-module $M$, for every $m \in 0^*_M$, and for every $n \geq 0$, the element $cx^n$ annihilates $1 \otimes m \in Rx^0 \otimes_R M = (R[x, f] \otimes_R M)_0$. 
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Hochster and Huneke proved that, if $R$ is reduced and excellent, then the concepts of test element for modules and test element for ideals coincide for $R$. 
The Hochster–Huneke Existence Theorem

Theorem (M. Hochster and C. Huneke, 1994). If $\mathcal{R}$ is a reduced algebra of finite type over an excellent local ring of characteristic $p$, then $\mathcal{R}$ has a test element.
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Theorem (M. Hochster and C. Huneke, 1994). If $R$ is a reduced algebra of finite type over an excellent local ring of characteristic $p$, then $R$ has a test element.

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In fact, if $c \in R^\circ$ is such that $R_c$ is regular, then some power of $c$ is a test element for $R$.

Indeed, if $c \in R^\circ$ is such that $R_c$ is Gorenstein and $F$-regular, then some power of $c$ is a test element for $R$. 
Big test elements

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A big test element for $R$ is a $c \in R^\circ$ such that, for every $R$-module $M$, for every $m \in 0^*_M$, and for every $n \geq 0$, the element $cx^n$ annihilates $1 \otimes m \in Rx^0 \otimes_R M = (R[x, f] \otimes_R M)_0$. 
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For example, if there is a Frobenius splitting for $R$, that is, a $\mathbb{Z}$-homomorphism $\phi : R \longrightarrow R$ such that $\phi(sr^p) = \phi(s)r \ \forall \ r, s \in R \text{ and } \phi(1) = 1$, then $R$ is $F$-pure.
Right and left $R[x, f]$-modules

Such a Frobenius splitting $\phi$ for $R$ yields a structure as right $R[x, f]$-module on $R$, with $rx = \phi(r) \forall r \in R$. 
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In fact, in the local $F$-finite case, a Frobenius splitting $\phi$ for $R$ as above leads to an $x$-torsion-free left $R[x, f]$-module structure on $E$. 
Modification of known examples

If \( L \) is a left \( R[y, f] \)-module (where \( y \) is a variable) and \( u \in R \),
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If $L$ is a left $R[y, f]$-module (where $y$ is a variable) and $u \in R$, then $L$ is a left $R[x, f]$-module via $xg = uyg$ for all $g \in L$. 
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This idea can be used to prove the following.
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This idea can be used to prove the following.

**Theorem.** Let $(R, \mathfrak{m})$ be local and $F$-pure. Then $E_R(R/\mathfrak{m})$ can be given the structure of an $x$-torsion-free left $R[x, f]$-module that extends its $R$-module structure.
Proof

($(R, \mathfrak{m})$ is $F$-pure.) Reduce to the case where $R$ is complete, and write $R = S/b$ where $(S, \mathfrak{n})$ is a complete regular local ring of characteristic $p$ and $0 \neq b \neq S$. 
Proof

\((\mathcal{R}, \mathfrak{m})\) is \(F\)-pure.) Reduce to the case where \( R \) is complete, and write \( R = S / \mathfrak{b} \) where \((S, \mathfrak{n})\) is a complete regular local ring of characteristic \( p \) and \( 0 \neq \mathfrak{b} \neq S \).

Set \( E := E_S(S/\mathfrak{n}) \) and note that \((0 :_E \mathfrak{b}) = E_R(R/\mathfrak{m})\).
Proof

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Set $E := E_S(S/n)$ and note that $(0 :_E b) = E_R(R/m)$.

Since $E \cong H_{n}^{\dim S}(S)$, there is a natural left $S[y, f]$-module structure on $E$ (where $y$ is a variable).
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The idea is to choose a \(u \in S\) so that, for the left \(S[x, f]\)-module structure on \(E\) for which \(xe = uye\) for all \(e \in E\), the subset \((0 :_E b)\) becomes an \(S[x, f]\)-submodule, and therefore a left \(R[x, f]\)-module.
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Set \(E := E_S(S/n)\) and note that \((0 :_E b) = E_R(R/m)\).

Since \(E \cong H^\dim S_n(S)\), there is a natural left \(S[y, f]\)-module structure on \(E\) (where \(y\) is a variable). The idea is to choose a \(u \in S\) so that, for the left \(S[x, f]\)-module structure on \(E\) for which \(xe =uye\) for all \(e \in E\), the subset \((0 :_E b)\) becomes an \(S[x, f]\)-submodule, and therefore a left \(R[x, f]\)-module. Moreover, we want to do this in such a way that \((0 :_E b)\) is \(x\)-torsion-free.
How do we choose \( u \)?

It turns out that \((0 :_E \mathfrak{b})\) is an \( \mathcal{S}[x, f] \)-submodule (for the structure in which \( xe = uye \) for all \( e \in E \)) if and only if \( u \in (\mathfrak{b}^p :_S \mathfrak{b}) \).
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It turns out that $(0 :_E \mathfrak{b})$ is an $S[x, f]$-submodule (for the structure in which $xe = uye$ for all $e \in E$) if and only if $u \in (\mathfrak{b}^p :_S \mathfrak{b})$. So we need to choose a good $u \in (\mathfrak{b}^p :_S \mathfrak{b})$. 
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Fedder’s Theorem (1983). (Recall that $\mathfrak{b}$ is a proper ideal of the complete regular local ring $(S, \mathfrak{n})$.) The ring $R = S/\mathfrak{b}$ is $F$-pure $\iff (\mathfrak{b}^p :_S \mathfrak{b}) \not\subseteq \mathfrak{n}^p$. 
How do we choose $u$?

It turns out that $(0 :_E \mathfrak{b})$ is an $S[x, f]$-submodule (for the structure in which $xe = uye$ for all $e \in E$) if and only if $u \in (\mathfrak{b}^p :_S \mathfrak{b})$. So we need to choose a good $u \in (\mathfrak{b}^p :_S \mathfrak{b})$. We use Fedder’s Theorem (1983). (Recall that $\mathfrak{b}$ is a proper ideal of the complete regular local ring $(S, \mathfrak{n})$.) The ring $R = S/\mathfrak{b}$ is $F$-pure $\iff (\mathfrak{b}^p :_S \mathfrak{b}) \subsetneq \mathfrak{n}^p$.

We choose $u \in (\mathfrak{b}^p :_S \mathfrak{b}) \setminus \mathfrak{n}^p$. This yields a left $R[x, f]$-module structure on $(0 :_E \mathfrak{b}) = E_R(R/m)$. 
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It turns out that \((0 :_E b)\) is an \( S[x, f]\)-submodule (for the structure in which \( xe = uye \) for all \( e \in E \)) if and only if \( u \in (b^{[p]} :_S b) \). So we need to choose a good \( u \in (b^{[p]} :_S b) \). We use

Fedder’s Theorem (1983). (Recall that \( b \) is a proper ideal of the complete regular local ring \((S, n)\).) The ring \( R = S/b \) is \( F \)-pure \( \iff (b^{[p]} :_S b) \not\subseteq n^{[p]} \).

We choose \( u \in (b^{[p]} :_S b) \setminus n^{[p]} \). This yields a left \( R[x, f] \)-module structure on \((0 :_E b) = E_R(R/m)\).

The choice of \( u \) outside \( p^{[p]} \) for every prime ideal \( p \) of \( S \) ensures that \( E_R(R/m) \) is \( x \)-torsion-free. \( \Box \)
A strategy

Suppose that \((R, m)\) is local and \(E := E_R(R/m)\) has a structure as an \(x\)-torsion-free left \(R[x, f]\)-module that extends its \(R\)-module structure.
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In order to draw conclusions about the existence of big test elements for \(R\), we are going to consider an arbitrary \(R\)-module \(M\), and embed (homogeneously, and over \(R[x, f]\)) the graded left \(R[x, f]\)-module \(R[x, f] \otimes_R M\) into an \(x\)-torsion-free graded left \(R[x, f]\)-module \(K\) that has some properties in common with \(E\).
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The theory of graded annihilators, \(E\)-special \(R\)-ideals and special annihilator submodules will help with this.
Special annihilator submodules

The graded two-sided ideals of $R[x, f]$ are precisely the subsets of the form $\bigoplus_{n \geq 0} a_n x^n$, where

$$a_0 \subseteq a_1 \subseteq \cdots \subseteq a_n \subseteq \cdots$$

is an ascending (and so ultimately stationary) sequence of ideals of $R$. 
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An $R[x, f]$-submodule of a left $R[x, f]$-module $H$ is said to be a special annihilator submodule of $H$ if it has the form

$$\text{ann}_H(\mathcal{A}) = \{ h \in H : \theta h = 0 \text{ for all } \theta \in \mathcal{A} \}$$

for some graded two-sided ideal $\mathcal{A}$ of $R[x, f]$. 

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We shall use $\mathcal{A}(H)$ to denote the set of special annihilator submodules of $H$. 
Graded annihilators
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The graded annihilator $\text{gr-ann}_{R[x,f]} H$ of a left $R[x,f]$-module $H$ is the largest graded two-sided ideal of $R[x,f]$ that annihilates $H$. 
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$$\text{gr-ann}_{R[x,f]} H = bR[x,f] = \bigoplus_{n \geq 0} bx^n$$

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The basic correspondence

Let $H$ be an $x$-torsion-free left $R[x, f]$-module.
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Let $H$ be an $x$-torsion-free left $R[x, f]$-module. There is an order-reversing bijection

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Let \( c \in R^\circ \) be such that \( cx^n g = 0 \ \forall \ n \geq n_0 \).
Proof of (iii) ⇒ (i)

Let $c \in R^\circ$ be such that $cx^ng = 0 \\forall \ n \geq n_0$.

Then $g \in \text{ann}_H(\bigoplus_{n\geq n_0} Rcx^n) \in A(H)$. Let $a \in \mathcal{I}(H)$ correspond to $\text{ann}_H(\bigoplus_{n\geq n_0} Rcx^n) \in A(H)$,
Proof of (iii) ⇒ (i)

Let $c \in R^o$ be such that $cx^n g = 0 \ \forall \ n \geq n_0$.

Then $g \in \text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in \mathcal{A}(H)$. Let $a \in \mathcal{I}(H)$ correspond to $\text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in \mathcal{A}(H)$, so that

$$aR[x, f] = \text{gr-ann}_{R[x, f]}(\text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n)).$$
Proof of \((iii) \Rightarrow (i)\)

Let \(c \in R^\circ\) be such that \(cx^ng = 0 \ \forall \ n \geq n_0\). Then \(g \in \text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in A(H)\). Let \(a \in I(H)\) correspond to \(\text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in A(H)\), so that

\[ aR[x, f] = \text{gr-ann}_{R[x,f]}(\text{ann}_H(\bigoplus_{n \geq n_0} Rcx^n)). \]

Since \(c \in a\), \(\text{ht} \ a \geq 1\), so that \(b \subseteq a\).
Proof of (iii) ⇒ (i)

Let $c \in R^\circ$ be such that $cx^ng = 0 \ \forall \ n \geq n_0$.

Then $g \in \operatorname{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in \mathcal{A}(H)$. Let $a \in \mathcal{I}(H)$ correspond to $\operatorname{ann}_H(\bigoplus_{n \geq n_0} Rcx^n) \in \mathcal{A}(H)$, so that

$$aR[x, f] = \operatorname{gr-ann}_{R[x, f]}(\operatorname{ann}_H(\bigoplus_{n \geq n_0} Rcx^n)).$$

Since $c \in a$, $\operatorname{ht} a \geq 1$, so that $b \subseteq a$. Therefore

$$g \in \operatorname{ann}_H(\bigoplus_{n \geq n_0} Rcx^n)$$

$$= \operatorname{ann}_H(aR[x, f]) \subseteq \operatorname{ann}_H(bR[x, f]). \quad \square$$
A strategy for \((R, \mathfrak{m})\) local & \(F\)-pure

Step 1. Set \(E = E_R(R/\mathfrak{m})\) and put an \(x\)-torsion-free left \(R[x, f]\)-module structure on \(E\).
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**Step 2.** Show that, for an arbitrary \(R\)-module \(M\), \(\exists\) an \(x\)-torsion-free graded left \(R[x, f]\)-module \(K(M)\) such that \(I(K(M)) = I(E)\), finite (so that \(\mathfrak{b}(K(M)) = \mathfrak{b}\)), and a homogeneous \(R[x, f]\)-monomorphism \(R[x, f] \otimes_R M \rightarrow K(M)\).
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**Step 3.** Deduce that, if \(m \in 0^*_M\), so that \(\exists\ c \in R^\circ\) such that \(cx^n(1 \otimes m) = 0\) in \(R[x, f] \otimes_R M\) for all \(n > > 0\),
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The graded companion

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The graded companion

For Step 2, starting with an $x$-torsion-free left $R[x, f]$-module structure on $E$, we shall construct from $E$ various $x$-torsion-free graded left $R[x, f]$-modules $L$ with $\mathcal{I}(L) = \mathcal{I}(E)$. 
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For all $n \geq 0$, set $E_n := E$. Then the graded companion of $E$ is the left $R[x, f]$-module $	ilde{E} := \bigoplus_{n \geq 0} E_n$, where the result of multiplying $h_n \in E_n = E$ on the left by $x$ is the element $xh_n \in E_{n+1} = E$. 
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For each graded two-sided ideal $\mathfrak{B}$ of $R[x, f]$, we have $\text{ann}_{\tilde{E}} \mathfrak{B} = \text{ann}_E \mathfrak{B}$. 
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For Step 2, starting with an \( x \)-torsion-free left \( R[x, f] \)-module structure on \( E \), we shall construct from \( E \) various \( x \)-torsion-free graded left \( R[x, f] \)-modules \( L \) with \( \mathcal{I}(L) = \mathcal{I}(E) \).

For all \( n \geq 0 \), set \( E_n := E \). Then the graded companion of \( E \) is the left \( R[x, f] \)-module \( \tilde{E} := \bigoplus_{n \geq 0} E_n \), where the result of multiplying \( h_n \in E_n = E \) on the left by \( x \) is the element \( xh_n \in E_{n+1} = E \).

For each graded two-sided ideal \( \mathfrak{B} \) of \( R[x, f] \), we have \( \text{ann}_{\tilde{E}} \mathfrak{B} = \text{ann}_E \mathfrak{B} \). Thus (\( \tilde{E} \) is \( x \)-torsion-free and) \( \mathcal{I}(\tilde{E}) = \mathcal{I}(E) \).
**Graded products**

Let \( \left( H^{(\lambda)} = \bigoplus_{n \in \mathbb{Z}} H_n^{(\lambda)} \right)_{\lambda \in \Lambda} \) be a non-empty family of \( \mathbb{Z} \)-graded left \( R[x, f] \)-modules.
Graded products

Let \( \left( H^{(\lambda)} = \bigoplus_{n \in \mathbb{Z}} H^{(\lambda)}_n \right) \) be a non-empty family of \( \mathbb{Z} \)-graded left \( R[x, f] \)-modules.

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has a natural structure as a (\( \mathbb{Z} \)-graded) left \( R[x, f] \)-module in which

\[
x(h_n^{(\lambda)})_{\lambda \in \Lambda} = (x h_n^{(\lambda)})_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} H_{n+1}^{(\lambda)}
\]

for all \( (h_n^{(\lambda)})_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} H_n^{(\lambda)} \).
Graded products, continued

Since $H$ is the product of $(H^{(\lambda)})_{\lambda \in \Lambda}$ in the category of $\mathbb{Z}$-graded left $R[x, f]$-modules and homogeneous $R[x, f]$-homomorphisms (of degree 0), we shall denote the module $H$ by $\prod_{\lambda \in \Lambda}^{' } H^{(\lambda)}$, and refer to it as the graded product of the $H^{(\lambda)}$. 
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If $H^{(\lambda)}$ is $x$-torsion-free for all $\lambda \in \Lambda$, then $\prod'_{\lambda \in \Lambda} H^{(\lambda)}$ is also $x$-torsion-free.
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If $H^{(\lambda)}$ is $x$-torsion-free for all $\lambda \in \Lambda$, then $\prod'_{\lambda \in \Lambda} H^{(\lambda)}$ is also $x$-torsion-free. In that $x$-torsion-free case, if $\mathcal{I}(H^{(\lambda)}) = \mathcal{I}(E)$ for all $\lambda \in \Lambda$, then $\mathcal{I} \left( \prod'_{\lambda \in \Lambda} H^{(\lambda)} \right) = \mathcal{I}(E)$ also.
Extensions

Let \( b \in \mathbb{Z} \) and \( W = \bigoplus_{n \geq b} W_n \) be a \( \mathbb{Z} \)-graded left \( R[x, f] \)-module; let \( (g_i)_{i \in I} \) be a family of arbitrary elements of \( W_b \).
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\[
\begin{align*}
  f : & V & \longrightarrow & V \\
  \quad (r_i)_{i \in I} & \longmapsto (r_i^p)_{i \in I}
\end{align*}
\]

be the Frobenius map.
Extensions

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$$f : V \rightarrow V \begin{cases} (r_i)_{i \in I} & \mapsto (r_i^p)_{i \in I} \end{cases}$$

be the Frobenius map. Set

$$K := \left\{ (r_i)_{i \in I} \in V : \sum_{i \in I} r_i g_i = 0 \right\},$$

an $R$-submodule of $V$. 
Extensions, continued

Let $h \in \mathbb{Z}$ with $h > 0$. 
Extensions, continued

Let $h \in \mathbb{Z}$ with $h > 0$. Then the graded left $R[x, f]$-module

$$(V/f^{-h}(K)) \oplus \cdots \oplus (V/f^{-1}(K)) \oplus W_b \oplus W_{b+1} \oplus \cdots,$$
Extensions, continued

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which has $W$ as a graded $R[x, f]$-submodule and is such that (for all $v = (r_i)_{i \in I} \in V$)

$$x(v + f^{-j}(K)) = \begin{cases} 
  f(v) + f^{-(j-1)}(K) & \text{if } h \geq j \geq 2, \\
  \sum_{i \in I} r_i g_i & \text{if } j = 1,
\end{cases}$$
Extensions, continued

Let $h \in \mathbb{Z}$ with $h > 0$. Then the graded left $R[x, f]$-module

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is called the $h$-place extension of $W$ by $(g_i)_{i \in I}$, and denoted by $\text{exten}(W; (g_i)_{i \in I}; h)$. 
Extensions, continued

The action of $x$ is such that, if $w'_j \in V / f^{-j}(K)$ with $h \geq j \geq 1$, then $xw'_j = 0 \iff w'_j = 0$. 
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Extensions, continued

The action of $x$ is such that, if $w_j' \in V/f^{-j}(K)$ with $h \geq j \geq 1$, then $xw_j' = 0 \iff w_j' = 0$. This has the consequences that, if $W$ is $x$-torsion-free, then so too is $\text{exten}(W; (g_i)_{i \in I}; h)$, and, in that $x$-torsion-free case, $\mathcal{I}(W) = \mathcal{I}(\text{exten}(W; (g_i)_{i \in I}; h))$,.
Extensions, continued

The action of $x$ is such that, if $w'_j \in V/f^{-j}(K)$ with $h \geq j \geq 1$, then $xw'_j = 0 \iff w'_j = 0$. This has the consequences that, if $W$ is $x$-torsion-free, then so too is $\text{exten}(W; (g_i)_{i \in I}; h)$, and, in that $x$-torsion-free case, $I(W) = I(\text{exten}(W; (g_i)_{i \in I}; h))$, so that if $I(W) = I(E)$, then $I(\text{exten}(W; (g_i)_{i \in I}; h)) = I(E)$ too.
Preservation of $x$-torsion-freeness & $\mathcal{I}(\bullet)$

We have now seen that the properties of being $x$-torsion-free and having set of special $R$-ideals equal to the set of $E$-special $R$-ideals $\mathcal{I}(E)$ are preserved under the operations of
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- passing to graded companions,
- taking graded products, and
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they are also preserved under

- shifting.
The Embedding Theorem

Theorem. Let $E$ be an injective cogenerator of $R$ with a left $R[x, f]$-module structure.
The Embedding Theorem

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Theorem. Let $E$ be an injective cogenerator of $R$ with a left $R[x, f]$-module structure. Let $M$ be an $R$-module. Then there is a left $R[x, f]$-module $K(M)$, graded by the set $\mathbb{N}_0$ of non-negative integers, formed as the graded product of a family of extensions of shifts of graded products of copies of the graded companion $\tilde{E}$ of $E$,
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\nu : R[x, f] \otimes_R M = \bigoplus_{n \geq 0} (Rx^n \otimes_R M) \longrightarrow K(M).
$$
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$$\nu : R[x, f] \otimes_R M = \bigoplus_{n \geq 0} (Rx^n \otimes_R M) \longrightarrow K(M).$$

If $E$ is $x$-torsion-free, then so too are $K(M)$ and $R[x, f] \otimes_R M$, and then

$$\mathcal{I}(R[x, f] \otimes_R M) \subseteq \mathcal{I}(K(M)) = \mathcal{I}(E) \quad \forall \ M.$$
Some applications of the Embedding Theorem

Theorem. Suppose that \((R, \mathfrak{m})\) is local. Then \(R\) is \(F\)-pure if and only if the \(R\)-module structure on \(E_R(R/\mathfrak{m})\) can be extended to an \(x\)-torsion-free left \(R[x, f]\)-module structure.
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Theorem. Suppose that \((R, \mathfrak{m})\) is local and \(F\)-pure. Then \(R\) has a big tight closure test element, even if it is not excellent.

Theorem. \((R\) is not assumed to be local here.) Suppose that \(R\) is excellent and \(F\)-pure. Then \(R\) has a big tight closure test element.
Some applications of the Embedding Theorem

Theorem. Suppose that \((R, m)\) is local. Then \(R\) is \(F\)-pure if and only if the \(R\)-module structure on \(E_R(R/m)\) can be extended to an \(x\)-torsion-free left \(R[x, f]\)-module structure.

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Theorem. \((R\) is not assumed to be local here.) Suppose that \(R\) is excellent and \(F\)-pure. Then \(R\) has a big tight closure test element. In fact, if \(c \in R^\circ\) is such that \(R_c\) is regular, then \(c\) itself is a big test element for \(R\).
The non-$F$-pure case

The methods described in this talk are particularly well suited for use when $R$ is $F$-pure, because then there are many naturally occurring $x$-torsion-free left $R[x, f]$-modules.
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However, the methods can be refined for use in some non-$F$-pure cases.
The non-$F$-pure case

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However, the methods can be refined for use in some non-$F$-pure cases. For example, when $(R, m)$ is complete, local and reduced, it is possible to put a left $R[x, f]$-module structure on $E := E_R(R/m)$ that is sufficiently non-trivial so that $\text{ht}(0 :_R \Gamma_x(E)) > 0$. 
The non-$F$-pure case

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One can use such an $R[x, f]$-module structure on $E$, in conjunction with the Embedding Theorem, to prove an existence theorem for big test elements when $R$ is reduced, excellent and local.
More applications of the Embedding Theorem

Theorem. Suppose that \((R, m)\) is local, excellent and reduced. Then \(R\) has a big test element. In fact, \(\exists n > 0\) such that, \(\forall c \in R^o\) for which \(R_c\) is regular, \(c^n\) is a big test element for \(R\).
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Theorem. (\(R\) is not assumed to be local here.) Each big test element \(c\) for \(R\) is automatically locally stable, that is \(c/1 \in R_p\) is a big test element for \(R_p\) for all \(p \in \text{Spec}(R)\).
More applications of the Embedding Theorem

Theorem. Suppose that \((R, m)\) is local, excellent and reduced. Then \(R\) has a big test element. In fact, \(\exists n > 0\) such that, \(\forall c \in \mathcal{R}_n\) for which \(\mathcal{R}_c\) is regular, \(c^n\) is a big test element for \(R\).

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Theorem. Assume that \(R\) is excellent (but not necessarily local). Then each big test element \(c\) for \(R\) is automatically completely stable, that is \(c/1 \in \mathcal{R}_p\) is a big test element for \(\mathcal{R}_p\) for all \(p \in \text{Spec}(R)\).
Proof of the Embedding Theorem

Begin with an injective cogenerator $E$ of $R$ with an $R[x, f]$-module structure.
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Begin with an injective cogenerator $E$ of $R$ with an $R[x, f]$-module structure.

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Proof the Embedding Theorem, continued

For each $j \in J$, set $H^{(j)} := \tilde{E}$;
Proof the Embedding Theorem, continued

For each \( j \in J \), set \( H^{(j)} := \tilde{E} \); set \( L^{(0)} := \prod'_{j \in J} H^{(j)} \), so that \( (L^{(0)})_0 = \prod_{j \in J} E^{(j)} \).
Proof the Embedding Theorem, continued

For each \( j \in J \), set \( H^{(j)} := \tilde{E} \); set \( L^{(0)} := \prod'_{j \in J} H^{(j)} \), so that \( (L^{(0)})_0 = \prod_{j \in J} E^{(j)} \). Identify \( M \) with \( Rx^0 \otimes_R M \).
Proof the Embedding Theorem, continued

For each $j \in J$, set $H^{(j)} := \tilde{E}$; set $L^{(0)} := \prod'_{j \in J} H^{(j)}$, so that $(L^{(0)})_0 = \prod_{j \in J} E^{(j)}$. Identify $M$ with $R x^0 \otimes_R M$.

We can then define, for each $n > 0$, an $R$-hom.

$(\lambda^{(0)})_n : R x^n \otimes_R M \longrightarrow L^{(0)}_n$ for which

$$(\lambda^{(0)})_n(r x^n \otimes m) = r x^n (\lambda^{(0)})_0(m) \forall r \in R, m \in M.$$
Proof the Embedding Theorem, continued

For each $j \in J$, set $H^{(j)} := \tilde{E}$; set $L^{(0)} := \prod'_{j \in J} H^{(j)}$, so that $(L^{(0)})_0 = \prod_{j \in J} E^{(j)}$. Identify $M$ with $R x^0 \otimes_R M$.

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$$(\lambda^{(0)})_n(r x^n \otimes m) = r x^n (\lambda^{(0)})_0(m) \ \forall \ r \in R, m \in M.$$ Then

$$\bigoplus_{i \geq 0} (\lambda^{(0)})_i : \bigoplus_{i \geq 0} (Rx^i \otimes_R M) \longrightarrow \prod'_{j \in J} H^{(j)}$$
Proof the Embedding Theorem, continued

For each \( j \in J \), set \( H^{(j)} := \tilde{E} \); set \( L^{(0)} := \prod'_{j \in J} H^{(j)} \), so that \( (L^{(0)})_0 = \prod_{j \in J} E^{(j)} \). Identify \( M \) with \( Rx^0 \otimes_R M \).

We can then define, for each \( n > 0 \), an \( R \)-hom. \((\lambda^{(0)})_n : Rx^n \otimes_R M \rightarrow L^{(0)}_n \) for which
\[(\lambda^{(0)})_n(rx^n \otimes m) = rx^n (\lambda^{(0)})_0(m) \quad \forall r \in R, m \in M.\]

Then
\[
\bigoplus_{i \geq 0} (\lambda^{(0)})_i : \bigoplus_{i \geq 0} (Rx^i \otimes_R M) \rightarrow \prod'_{j \in J} H^{(j)}
\]

is a homogeneous \( R[x, f] \)-homomorphism \( \lambda^{(0)} : R[x, f] \otimes_R M \rightarrow L^{(0)} \) for which \( (\lambda^{(0)})_0 \) is a monomorphism.
The \((\lambda(0))^n\) \((n > 0)\) might not be monomorphic!

Choose \(n > 0\), and apply the last slide to \(R x^n \otimes_R M\):
The \((\lambda^{(0)})_n \ (n > 0)\) might not be monomorphic!

Choose \(n > 0\), and apply the last slide to \(Rx^n \otimes_R M\): there is a family \((G^{(j,n)})_{j \in Y_n}\) of graded left \(R[x, f]\)-modules, all equal to \(\tilde{E}\), and a homogeneous \(R[x, f]\)-homomorphism

\[
R[x, f] \otimes_R (Rx^n \otimes_R M) \longrightarrow \prod_{j \in Y_n} \ 'G^{(j,n)}
\]

which is monomorphomic in degree 0.
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\(R[x, f]\)-modules, all equal to \(\tilde{E}\), and a homogeneous

\(R[x, f]\)-homomorphism

\[
R[x, f] \otimes_R (R x^n \otimes_R M) \longrightarrow \prod_{j \in Y_n} \ ' G^{(j,n)}
\]

which is monomorphemic in degree 0. Apply the shift

\((\bullet)(-n)\): we get a homogeneous \(R[x, f]\)-hom.

\[
\zeta^{(n)} : \bigoplus_{j \geq n} (R x^j \otimes_R M) \longrightarrow \left( \prod_{j \in Y_n} \ ' G^{(j,n)} \right)(-n) =: Q^{(n)}
\]

which is monomorphemic in degree \(n\).
But we need a map from $\bigoplus_{j \geq 0}(Rx^j \otimes_R M)$!

Let $\{m_i : i \in I\}$ be a generating set for $M$;
But we need a map from $\bigoplus_{j \geq 0}(Rx^j \otimes_R M)$!

Let \( \{m_i : i \in I\} \) be a generating set for \( M \); set
\[
g_i = (\zeta^{(n)})_n(x^n \otimes m_i) \forall i \in I.
\]
But we need a map from $\bigoplus_{j\geq 0}(Rx^j \otimes_R M)$!

Let $\{m_i : i \in I\}$ be a generating set for $M$; set $g_i = (\zeta^{(n)})_n(x^n \otimes m_i) \forall i \in I$. One can extend $\zeta^{(n)}$ to a homogeneous $R[x,f]$-homomorphism

$$\lambda^{(n)} : \bigoplus_{j\geq 0}(Rx^j \otimes_R M) \longrightarrow \text{exten}(Q^{(n)}; (g_i)_{i\in I}; n) =: L^{(n)},$$

such that $\lambda^{(n)}$ is monomorphic in degree $n$. 
But we need a map from $\bigoplus_{j \geq 0} (Rx^j \otimes_R M)$!

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such that $\lambda^{(n)}$ is monomorphically in degree $n$. Note that, if $E$ is $x$-torsion-free, then so too is $L^{(n)}$ and $\mathcal{I}(L^{(n)}) = \mathcal{I}(E) \forall n \geq 0$. 
Use the graded product of the $L^{(n)}$ ($n \geq 0$)

There is a homogeneous $R[x, f]$-homomorphism

$$\nu = \bigoplus_{j \geq 0} \nu_j : R[x, f] \otimes_R M \longrightarrow \prod_{n \geq 0} \overset{'}{L}^{(n)} =: K(M)$$

such that $\nu_j(\xi_j) = ((\lambda^{(n)})_j(\xi_j))_{n \geq 0}$ for all $j \geq 0$ and $\xi_j \in Rx^j \otimes_R M$. 
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Note that, if $E$ is $x$-torsion-free, then so too is $K(M)$ and $\mathcal{I}(K(M)) = \mathcal{I}(E)$. □